Random driving of a nonlinear oscillator

A. Gerasimov
Fermi National Accelerator Laboratory, P.O.Box 500, Batavia, Illinois 60510

S. Y. Lee
Physics Department, Indiana University, Bloomington, Indiana 47405
(Received 21 June 1993; revised manuscript received 7 December 1993)

The evolution equations for the density distribution and its two phase-space point correlation for nonlinear oscillators under the influence of an external random driving force are derived. We found that the correlation function obeys a Fokker-Planck-like evolution equation with decoherence, diffusion, and source terms. An asymptotic steady state solution is discussed, where the small-amplitude and short-wavelength spatial fluctuations of density ("microstructure") are found to be the special effect of the coherent random driving, distinguishing it from incoherent noise.

PACS number(s): 05.45.+b, 03.20.+i, 29.20.Dh

I. INTRODUCTION

The influence of random forces on nonlinear oscillators is a common problem in random processes theory, with many applications in all fields of science. For an individual particle experiencing a randomly applied force, the resulting particle motion has the characteristics of Brownian motion (see, e.g., [1]). Assuming that the probability function of finding a particle in the phase space is independent of the initial phase-space coordinates (Markovian process), the evolution equation of the particle distribution function is reduced to the Fokker-Planck equation.

A similar but not identical problem is the dynamics of the particle distribution of a nonlinear oscillator driven by a random force, which is the same for all particles. One example of such a situation is the effect of rf noise in high-energy particle accelerators [2–5]. Thus, one confronts a somewhat unusual problem of a random coherent driving force. The physics of this problem is quite distinct from the common incoherent-noise stochastic models and in our opinion deserves a study even apart from its possible particle accelerator applications.

For our random-driving system, the average (over the ensemble of driving forces) distribution function can be shown to satisfy the same Fokker-Planck evolution equation that appears when each particle is affected independently (incoherent noise). Indeed, for a small-amplitude time varying driving force, the response of the (periodic) trajectory of a nonlinear oscillator is determined by the spectral component of the force near the harmonics of the frequency of the oscillations. The important point is that the amplitudes of different harmonics of a sufficiently long section of random signal are statistically independent. Thus, to the zeroth-order approximation, the coherently random driving force and the Brownian motion with statistically independent random forces produce the same results. Many extensive studies have been published [2–4], where the Fokker-Planck equation was analyzed and solved using averaging techiques in the small noise and fast oscillation regime. The theory has been verified by numerical simulations [4] and has also been confirmed by experimental observations [5].

Beyond the zeroth-order approximation, how do the individual density distributions differ from the average one? What is the space time correlation in the density distribution function? Will the density fluctuations be smoothed out by the random noise? What is the effect of the nonlinearity on fluctuations? In the present paper, we address these questions by studying the fluctuations in the ensemble of density distributions, which can be described by a correlation function in both phase space and time, as is conventional in kinetic theories. The correlation function formalism is most adequate as it allows one to quantify the leading-order effect of the noise "coherence."

In this paper, we study the spatial spectrum of the fluctuations (same-time correlation function) in the limit of small noise and large nonlinearity. For accelerator physics, the practical importance of such density fluctuations is not fully clear at this time. The formalism of the present paper provides, however, the tools to study it in the future. It can be conjectured that many aspects of coherent instability dynamics, such as the source of short-wavelength perturbations (for microwave instability), Landau damping rates, etc., are affected by the density fluctuations. The fluctuations could be observable in the longitudinal Schottky spectra of the bunches [6]. A similar coherent-noise fluctuation contribution should be observable in the longitudinal Schottky spectrum of the coasting beam of stochastic cooling systems and may have an impact on their performance that is not appreciated at the present time.

The plan of the paper is as follows. A model of a nonlinear oscillator with random coherent driving force is introduced. After defining the correlation function, we obtain a self-contained description of fluctuations by deriving the evolution equation for the correlator. A solution of this evolution equation in the limit of small noise and fast oscillations is discussed in Sec. IV. The conclusion is given in Sec. V.

1063-651X/94/49(5)/3881(7)/$06.00

II. MODEL

We consider the general form of the Hamiltonian for a nonlinear oscillator with a random driving force,

$$H = \frac{p^2}{2} + g(q) + h(p,q)\xi(t), \tag{1}$$

where $g(q)$ is an arbitrary nonlinear potential and $\xi(t)$ is for simplicity, yet without loss of generality, chosen to be the white noise, i.e.,

$$\langle \xi(t)\xi(t')\rangle_{\{\xi\}} = \delta(t-t'), \tag{2}$$

where the ensemble average is denoted by $\langle\cdots\rangle_{\{\xi\}}$. For a rf voltage noise, $h(p,q) = g(q)$ and for the rf phase noise, $h(p,q) = p$. The exact form of the perturbation term $h(p,q)$ may affect the calculation of diffusion coefficients. However, the formalism to be discussed is independent of the form of the noise term.

In the absence of particle interactions (or collisions), the evolution of the density distribution is governed by the Vlasov equation,

$$\frac{\partial f}{\partial t} - \left(\frac{\partial g}{\partial q} + \frac{\partial h}{\partial q}\xi(t)\right)\frac{\partial f}{\partial p} + \left(p + \frac{\partial h}{\partial p}\xi(t)\right)\frac{\partial f}{\partial q} = 0. \tag{3}$$

In what follows, the Vlasov equation will be used instead of the stochastic equations of motion in order to derive the evolution equations for the statistical average of density function and correlation function, etc.

The statistical properties of the fluctuating quantity f are appropriately defined by the ensemble average of the distribution function,

$$\bar{f}(p,q,t) = \langle f(p,q,t)\rangle_{\{\xi\}}, \tag{4}$$

and the correlation function of the density fluctuations in adjacent phase-space points,

$$\begin{aligned} K(p,q,\tilde{p},\tilde{q},t) = \langle &[f(p,q,t) - \bar{f}(p,q,t)] \\ &\times[f(\tilde{p},\tilde{q},t) - \bar{f}(\tilde{p},\tilde{q},t)]\rangle_{\{\xi\}}. \end{aligned} \tag{5}$$

We limit ourselves by considering only the same-time correlator K and study therefore only the spatial, but not the time, correlation properties of the fluctuations.

Hereafter, we use the action-angle variables J, Ψ of the unperturbed $[h(p,q) = 0]$ Hamiltonian (1), which will be assumed to be known, to analyze these evolution equations. The perturbed Hamiltonian H in these variables has the form

$$H = H_0(J) + V(J,\Psi)\xi(t), \tag{6}$$

where $V(J,\Psi) = h(q(J,\Psi))$ and $H_0(J) = p^2/2 + g(q)$ are known functions.

III. EVOLUTION EQUATIONS

Both the average density $\bar{f}$ and the correlator K are evolving in time. We will derive the evolution equations for both quantities using basically the conventional techniques of the theory of stochastic differential equations [1]. It had been shown previously [2–4] that the evolution of the average density obeys the Fokker-Planck equation. However, to the authors' knowledge, the evolution of the density fluctuations has never been studied.

In the action-angle variables, the average density and the correlator are given by $\bar{f} = \bar{f}(J,\Psi,t)$ and $K = K(J,\Psi,\tilde{J},\tilde{\Psi},t)$. We will also use the notation $\tilde{\Psi} = \Psi + \varphi$ and compressed notations for the phase-space coordinates $(1) = x_{1i} = (J,\Psi)$ and $(2) = x_{2i} = (\tilde{J},\tilde{\Psi})$. Taking the infinitesimally small time increment Δt, one obtains the derivatives of the average density

$$\frac{\partial \bar{f}(1)}{\partial t} = \lim_{\Delta t\to 0}\frac{\Delta\bar{f}(1)}{\Delta t}, \tag{7}$$

and similarly the correlator

$$\begin{aligned} \frac{\partial K}{\partial t} = \lim_{\Delta t\to 0}\frac{1}{\Delta t}[&\langle\Delta f(1)f(2)\rangle + \langle f(1)\Delta f(2)\rangle \\ &+\langle\Delta f(1)\Delta f(2)\rangle - \bar{f}(1)\langle\Delta f(2)\rangle \\ &-\bar{f}(2)\langle\Delta f(1)\rangle - \langle\Delta f(1)\rangle\langle\Delta f(2)\rangle]. \end{aligned} \tag{8}$$

The increment of the density $\Delta f = f(t+\Delta t) - f(t)$ can be expressed, due to the conservation of the phase-space density, as the second-order Taylor expansion,

$$\Delta f = \frac{\partial f}{\partial x_i}\Delta x_i + \frac{1}{2}\frac{\partial^2 f}{\partial x_i \partial x_k}\Delta x_i \Delta x_k, \tag{9}$$

where a summation over repeating indices is assumed. The increments of the phase-space variables Δx_i in time Δt can be obtained from the stochastic equations of motion discussed in Appendix A. The second-order terms in Δx were kept because of the properties of the white noise, where the average of quadratic terms $\Delta x_i \Delta x_j$ produces terms linear in Δt. Any higher order terms result in higher than the first order in Δt contributions. The expansion (9) and the subsequent averaging procedures are the standard techniques in the Fokker-Planck equation derivation in stochastic analysis (see, e.g., [1]).

Making ensemble averages, one finds that the averages of products of x's and f's are factorizable, i.e.,

$$\begin{aligned} \langle\Delta f(1)\rangle = \langle\Delta x_i\rangle\left\langle\frac{\partial f(1)}{\partial x_{1i}}\right\rangle + \frac{1}{2}\langle\Delta x_{1i}\Delta x_{1k}\rangle \\ \times\left\langle\frac{\partial^2 f(1)}{\partial x_{1i}\partial x_{1k}}\right\rangle, \end{aligned} \tag{10}$$

$$\langle\Delta f(1)\Delta f(2)\rangle = \langle\Delta x_{1i}\Delta x_{2k}\rangle\left\langle\frac{\partial f(1)}{\partial x_{1i}}\frac{\partial f(2)}{\partial x_{2k}}\right\rangle, \tag{11}$$

etc. This is due to the fact that the increments Δx_i depend on the noise $\xi(t)$ only in the time range between t and $t + \Delta t$. Substituting Eqs. (10, 11) into Eqs. (7, 8), one obtains then the evolution equations for the moments of the density in the form

$$\frac{\partial \bar{f}}{\partial t} = \lim_{\Delta t\to 0}\frac{1}{\Delta t}\left\{\langle\Delta x_{1i}\rangle\frac{\partial\bar{f}}{\partial x_{1i}} + \frac{1}{2}\langle\Delta x_{1i}\Delta x_{1k}\rangle\frac{\partial^2\bar{f}}{\partial x_{1i}\partial x_{1k}}\right\} \tag{12}$$

$$\frac{\partial K}{\partial t} = \lim_{\Delta t\to 0}\frac{1}{\Delta t}\left\{\langle\Delta x_{1i}\rangle\frac{\partial K}{\partial x_{1i}} + \langle\Delta x_{2i}\rangle\frac{\partial K}{\partial x_{2i}} + \frac{1}{2}\langle\Delta x_{1i}\Delta x_{1k}\rangle\frac{\partial^2 K}{\partial x_{1i}\partial x_{1k}} + \frac{1}{2}\langle\Delta x_{2i}\Delta x_{2k}\rangle\frac{\partial^2 K}{\partial x_{2i}\partial x_{2k}} + \langle\Delta x_{1i}\Delta x_{2k}\rangle\left(\frac{\partial \bar f(1)}{\partial x_{1i}}\frac{\partial \bar f(2)}{\partial x_{2k}} + \frac{\partial^2 K}{\partial x_{1i}\partial x_{2k}}\right)\right\}. \quad (13)$$

The moments of Δx's that are present in Eqs. (12,13) can be computed quite straightforwardly by using the conventional techniques from the theory of stochastic differential equations. The details of this calculation are given in Appendix A. Combining Eq. (12) and the moments Eq. (A4) in the Vlasov equation Eq. (3), the evolution equation for the average $\bar f$ becomes the conventional Fokker-Planck equation,

$$\frac{\partial \bar f}{\partial t} = \left[\frac{1}{2}\left(\frac{\partial^2 V}{\partial J\partial\Psi}\frac{\partial V}{\partial J} - \frac{\partial^2 V}{\partial J^2}\frac{\partial V}{\partial \Psi}\right) + \omega(J)\right]\frac{\partial \bar f}{\partial \Psi} - \frac{1}{2}\left(\frac{\partial^2 V}{\partial \Psi^2}\frac{\partial V}{\partial J} - \frac{\partial^2 V}{\partial\Psi\partial J}\frac{\partial V}{\partial \Psi}\right)\frac{\partial \bar f}{\partial J} + \frac{1}{2}\left(\frac{\partial V}{\partial J}\right)^2\frac{\partial^2 \bar f}{\partial \Psi^2} + \frac{1}{2}\left(\frac{\partial V}{\partial \Psi}\right)^2\frac{\partial^2 \bar f}{\partial J^2} - \frac{\partial V}{\partial \Psi}\frac{\partial V}{\partial J}\frac{\partial^2 \bar f}{\partial\Psi\partial J}, \quad (14)$$

where $V = V(J,\Psi)$ and $\omega(J) = \frac{dH_0(J)}{dJ}$. For the correlator K, one obtains an evolution equation that is coupled to the mean $\bar f$,

$$\begin{aligned}\frac{\partial K}{\partial t} = &\left[\frac{1}{2}\left(\frac{\partial^2 V}{\partial J\partial\Psi}\frac{\partial V}{\partial J} - \frac{\partial^2 V}{\partial J^2}\frac{\partial V}{\partial \Psi}\right) + \omega(J)\right]\frac{\partial K}{\partial \Psi} - \frac{1}{2}\left(\frac{\partial^2 V}{\partial \Psi^2}\frac{\partial V}{\partial J} - \frac{\partial^2 V}{\partial\Psi\partial J}\frac{\partial V}{\partial \Psi}\right)\frac{\partial K}{\partial J}\\ &+\frac{1}{2}\left(\frac{\partial V}{\partial J}\right)^2\frac{\partial^2 K}{\partial \Psi^2} + \frac{1}{2}\left(\frac{\partial V}{\partial \Psi}\right)^2\frac{\partial^2 K}{\partial J^2} - \frac{\partial V}{\partial \Psi}\frac{\partial V}{\partial J}\frac{\partial^2 K}{\partial\Psi\partial J}\\ &+\left[\frac{1}{2}\left(\frac{\partial^2 \tilde V}{\partial \tilde J\partial\tilde\Psi}\frac{\partial \tilde V}{\partial \tilde J} - \frac{\partial^2 \tilde V}{\partial \tilde J^2}\frac{\partial \tilde V}{\partial \tilde\Psi}\right) + \omega(\tilde J)\right]\frac{\partial K}{\partial \tilde\Psi} - \frac{1}{2}\left(\frac{\partial^2 \tilde V}{\partial \tilde\Psi^2}\frac{\partial \tilde V}{\partial \tilde J} - \frac{\partial^2 \tilde V}{\partial\tilde\Psi\partial \tilde J}\frac{\partial \tilde V}{\partial \tilde\Psi}\right)\frac{\partial K}{\partial \tilde J}\\ &+\frac{1}{2}\left(\frac{\partial \tilde V}{\partial \tilde J}\right)^2\frac{\partial^2 K}{\partial \tilde\Psi^2} + \frac{1}{2}\left(\frac{\partial \tilde V}{\partial \tilde\Psi}\right)^2\frac{\partial^2 K}{\partial \tilde J^2} - \frac{\partial \tilde V}{\partial \tilde\Psi}\frac{\partial \tilde V}{\partial \tilde J}\frac{\partial^2 K}{\partial\tilde\Psi\partial \tilde J}\\ &+\frac{\partial V}{\partial \Psi}\frac{\partial \tilde V}{\partial \tilde\Psi}\frac{\partial^2 K}{\partial J\partial \tilde J} - \frac{\partial V}{\partial \Psi}\frac{\partial \tilde V}{\partial \tilde J}\frac{\partial^2 K}{\partial J\partial \tilde\Psi} - \frac{\partial V}{\partial J}\frac{\partial \tilde V}{\partial \tilde\Psi}\frac{\partial^2 K}{\partial \Psi\partial \tilde J} + \frac{\partial V}{\partial J}\frac{\partial \tilde V}{\partial \tilde J}\frac{\partial^2 K}{\partial \Psi\partial \tilde\Psi}\\ &+\frac{\partial V}{\partial \Psi}\frac{\partial \tilde V}{\partial \tilde\Psi}\frac{\partial \bar f(1)}{\partial J}\frac{\partial \bar f(2)}{\partial \tilde J} - \frac{\partial V}{\partial \Psi}\frac{\partial \tilde V}{\partial \tilde J}\frac{\partial \bar f(1)}{\partial J}\frac{\partial \bar f(2)}{\partial \tilde \Psi} - \frac{\partial V}{\partial J}\frac{\partial \tilde V}{\partial \tilde\Psi}\frac{\partial \bar f(1)}{\partial \Psi}\frac{\partial \bar f(2)}{\partial \tilde J} + \frac{\partial V}{\partial J}\frac{\partial \tilde V}{\partial \tilde J}\frac{\partial \bar f(1)}{\partial \Psi}\frac{\partial \bar f(2)}{\partial \tilde \Psi},\end{aligned} \quad (15)$$

where $\tilde V = V(\tilde J,\tilde\Psi)$.

IV. SMALL NOISE AND FAST OSCILLATION REGIME

A. Averaged evolution equations

On a long time scale, and in the small noise and fast oscillation regime, one can average the dependence of all quantities along the unperturbed trajectories $J = \text{const}$, $\Psi = \omega(J)t$. This approximation is well known under the name of "averaging of fast-oscillating variables" in the theory of Fokker-Planck equations (see, e.g., [1]), and was also used in previous studies of the average density diffusion [2–4]. For the Fokker-Planck Eq. (14), the procedure is technically very simple. One assumes that the density $\bar f$ is independent of Ψ by taking both the averages over ξ and Ψ (with the double average notation $\langle\langle ...\rangle\rangle$ for all coefficients. The resulting averaged Fokker-Planck equation becomes the well known diffusion equation [2–4],

$$\frac{\partial \bar f}{\partial t} = \frac{\partial}{\partial J}\left(D_J(J)\frac{\partial \bar f}{\partial J}\right), \quad (16)$$

where the diffusion intensity D_J is given by (see Appendix A for details)

$$D_J(J) = \frac{1}{2}\left\langle\left\langle\frac{(\Delta J)^2}{\Delta t}\right\rangle\right\rangle = \frac{1}{2}\sum_n n^2|V_n|^2, \quad (17)$$

where $V_n(J)$ are the harmonic amplitudes in the Fourier expansion of $V(J,\Psi)$ in the 2π periodic variable Ψ. This form of Fokker-Planck equation was verified experimentally [2].

The implementation of the same small noise and fast oscillations approximation in the evolution Eq. (15) for the correlator is somewhat more subtle. We will postulate at this point (and confirm it by the final results) that the correlator K does not depend on the phase Ψ but retains the dependence on the phase difference $\varphi = \Psi - \tilde\Psi$. The procedure for the evolution equation derivation then parallels that for the Fokker-Planck equation: one adds an extra averaging in Ψ in all moments in Eq. (13) while retaining the dependence on φ, and assumes K to depend on the phases Ψ and $\tilde\Psi$ only through the combination

$\varphi = \Psi - \tilde{\Psi}$. Using the moments calculated in Appendix A in the formulas (A6) and (A7), one obtains thus

$$\frac{\partial K}{\partial t} = \left(\omega(J) - \omega(\tilde{J})\right)\frac{\partial K}{\partial \varphi} + \frac{\partial}{\partial J}\left(D_J(J)\frac{\partial K}{\partial J}\right) + \frac{\partial}{\partial \tilde{J}}\left(D_J(\tilde{J})\frac{\partial K}{\partial \tilde{J}}\right) + G_\Psi(J,\tilde{J},\varphi)\frac{\partial^2 K}{\partial \varphi^2} + F_J(J,\tilde{J},\varphi)\left(\frac{\partial \bar{f}(J)}{\partial J}\frac{\partial \bar{f}(\tilde{J})}{\partial \tilde{J}} + \frac{\partial^2 K}{\partial J \partial \tilde{J}}\right), \quad (18)$$

where the functions F_J, G_Ψ appear from the moments in (A6), (A7),

$$G_\Psi(J,\tilde{J},\varphi) = \frac{1}{2}\sum_n\left(\left|\frac{\partial V_n}{\partial J}\right|^2 + \left|\frac{\partial \tilde{V}_n}{\partial \tilde{J}}\right|^2\right) + \sum_n \frac{\partial V_n(J)}{\partial J}\frac{\partial \tilde{V}_{-n}(\tilde{J})}{\partial \tilde{J}} e^{in\varphi},$$

$$F_J(J,\tilde{J},\varphi) = \sum_n n^2 V_n(J)\tilde{V}_{-n}(\tilde{J})e^{in\varphi}. \quad (19)$$

In the right-hand side of Eq. (18), the first term describes the decoherence due to the amplitude dependent frequency, while the remaining terms depict the Fokker-Planck-like diffusion due to the noise, and a source term. It is particularly interesting to note that the evolution equation of the correlator K contains an inhomogeneous source term, which depends on the gradient of the average distribution function. This means that the noise generates the fluctuations of density by shaking the distribution as a whole (hence the proportionality to $\frac{\partial \bar{f}}{\partial J}$). In the following, we will discuss the solutions of Eq. (18) in the small noise limit.

B. Asymptotic solution for the correlator

In the absence of noise, the solution of Eq. (18) for the correlator is trivial as only the first term in the right-hand side survives. The correlation "decays" or rather "decoheres" due to the phase mixing. Indeed, the general solution is an arbitrary function of $J, \tilde{J}$ and $\varphi + [\omega(J) - \omega(\tilde{J})]t$, so that for large enough time the correlation function becomes a fast-oscillating function of $J, \tilde{J}$. The time scale of this decoherence is $\tau_c \sim 1/\alpha\sigma_J$, where $\alpha = |\frac{d\omega}{dJ}|$ and σ_J is the rms value of J for the distribution $\bar{f}$. In the high-energy accelerators, processes such as beam decoherence after a longitudinal kick, filamentation following a misinjection, or Landau damping of coherent instabilities are related to this phenomenon.

In the presence of noise, the diffusion coefficient is $D_J \sim |V|^2$, and the characteristic diffusion time is $\tau_d \sim \sigma_J^2/D_J$. In the limit of small noise, the decoherence time is much shorter than the diffusion time. Furthermore, the correlation "injection," that is provided by the inhomogeneous term in Eq. (18), varies only on the slow time scale of diffusion. As a result, a quasistationary equilibrium correlation density will be established due to a balance between the slowly changing "injection" of correlations and their fast decay.

The shortness of the decoherence time manifests itself as well in the separation of the "spatial" (variables $J, \tilde{J}$) scales of the correlations and the density $\bar{f}$. One can visualize the dynamic process of the generation of correlations as follows. Correlations are injected first at the long wavelength $\lambda \sim \sigma_J$ by the source term, which arises from the perturbation to the distribution, shaken as a whole by noise. The "decoherence" process transports the correlations from long- to short-wavelength region. The small diffusion in action J is the essential dissipative mechanism at short wavelengths, leading to the decay of correlations on the long time scale. On the other hand, the phase diffusion, described by the second derivatives in φ in the evolution equation, is overshadowed by the fast "decoherence." Our discussion below is based on this cascade scenario of different time scales.

To analyze the quasistationary solution, we drop the time derivative of K and the phase diffusion terms in Eq. (18) . Another simplification comes from noticing that the correlator K is sharply peaked at the small distances $q = J - \tilde{J}$, where the "decoherence" term [first term in the right-hand side of Eq. (18)] is small. Assuming that $|q| \ll \sigma_J$, expanding all the coefficients in Eq. (18) to the leading order in q and keeping only the dominant derivatives in q, we obtain

$$\alpha q \frac{\partial K}{\partial \varphi} + 2D\frac{\partial^2 K}{\partial q^2} + F(\varphi)\left[c - \frac{\partial^2 K}{\partial q^2}\right] = 0, \quad (20)$$

where the quantities

$$\alpha = \frac{d\omega(J)}{dJ}, \quad D = D_J(J), \quad F(\varphi) = F_J(J,J,\varphi),$$

and

$$c = \left(\frac{\partial \bar{f}(J)}{\partial J}\right)^2$$

depend on J as a parameter.

We introduce now the Fourier spectrum $\tilde{K}(k,\varphi)$ of the correlator $K(q,\varphi)$,

$$\tilde{K}(k,\varphi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dq\, K(q,\varphi)e^{ikq}.$$

When transformed to the wave vector domain, the solution of Eq. (20) is given by (see Appendix B)

$$\tilde{K}(k,\varphi) = \begin{cases} 0, & \text{if } nk\alpha < 0, \\ \sum_{n\neq 0} B_n \tilde{T}_n(\varphi)e^{-\frac{2Dk^3}{3n\alpha}}, & \text{otherwise,} \end{cases} \quad (21)$$

where the functions $\tilde{T}_n(\varphi)$ are defined as

$$\tilde{T}_n(\varphi) = \exp\left\{-in\left(\varphi - \frac{1}{2D}\int_0^\varphi d\varphi_1\, F(\varphi_1)\right)\right\} \quad (22)$$

and the coefficients B_n as

$$B_n = \frac{c}{2\pi n \alpha} \int_0^{2\pi} d\varphi\, F(\varphi) \times \exp\left\{ in \left(\varphi - \frac{1}{2D} \int_0^{\varphi} d\varphi_1\, F(\varphi_1) \right) \right\}. \quad (23)$$

The cofficient B_0 is zero.

The dependence of the correlator spectrum on the phase φ is presented in the expression of Eq. (21) as an expansion in the set of functions $\tilde{T}_n(\varphi)$ of Eq. (22) rather than the conventional Fourier harmonics because of the phase dependence of the diffusion intensity in Eq. (20), which couples different harmonics.

The "correlation radius" of fluctuations q_c (the inverse characteristic wavelength) of the respective component n is seen to be $q_c \sim \left(\frac{2D_J}{|3n\alpha|}\right)^{1/3} \ll \sigma_J$. A special feature of the spectrum $\tilde{K}$ in Eq. (21) is its discontinuity. It is easy to see that this discontinuity is the manifestation of the long $\sim 1/|q|$ "tail" of the correlator K. Indeed, in the limit of small noise and large nonlinearity with $|q| \gg q_c$, both "diffusion" terms in Eq. (20) (proportional to the second derivative in q) are much smaller than the first term. Therefore, the solution of Eq. (20) in this region is given by

$$K(q,\varphi) = \frac{c}{\alpha q} \int_0^{\varphi} d\varphi_1\, F(\varphi_1).$$

Thus its Fourier spectrum $\tilde{K}(k,\varphi)$ has a step-function-like dependence on k.

It is possible to obtain a more general expression for the "tail" of K for $|q| \gg q_c$ that is not limited by the condition $|q| \ll \sigma_J$ by keeping the same terms of the primary evolution Eq. (18) (i.e., the first term in the right-hand side and the inhomogeneous term) without expanding in q. The resulting expression for the "tail" is

$$K(J,\tilde{J},\varphi,t) = \frac{1}{[\omega(J) - \omega(\tilde{J})]} \frac{\partial \bar{f}(J,t)}{\partial J} \frac{\bar{f}(\tilde{J},t)}{\partial \tilde{J}} \times \int_0^{\varphi} d\varphi_1\, F_J(J,\tilde{J},\varphi_1). \quad (24)$$

The intensity of the fluctuations is characterized by the rms amplitude P, defined as the value of the correlator K at $q = 0, \varphi = 0$. This can be calculated by integrating the spectrum $\tilde{K}$, i.e., $P = \int dk\, \tilde{K}(k,\varphi = 0)$. From Eq. (21), we obtain

$$P = \frac{\Gamma(1/3)}{(18)^{1/3}[\alpha(J)]^{2/3} D_J^{1/3}(J)} \sum_n \frac{B_n}{n^{2/3}}. \quad (25)$$

Thus, the fluctuation intensity is of the order $P \sim (D/\alpha)^{2/3}$ (recall that $B_n \sim |V_n|^2 \sim D$) and will be small for small noise and large nonlinearity.

V. CONCLUSIONS

We derived the evolution equation for the same-time correlation function of the density distribution fluctuations in the phase space of a nonlinear oscillator under the influence of coherent (same for all particles) random driving force. This driving produces both a diffusion in the amplitude of oscillation of particles and a continuous excitation of the phase-dependent density perturbations. We found that the two phase-space point correlation function satisfies a Fokker-Planck-like diffusion equation with a decoherence term and a source term. The source term is proportional to the product of the diffusion intensity and the gradient of the average distribution function, which satisfies the Fokker-Planck diffusion equation. In the limit of weak noise with large nonlinearity, the fluctuations of the correlation function in the action variable are characterized as small and short-ranged "microstructure" on top of a smooth mean distribution function.

We found that the phase mixing ("decoherence") of density perturbations due to the amplitude-dependent frequency of oscillations plays a major role in the dissipation mechanism of the fluctuations. A cascade-type solution was found for the case of small noise and large nonlinearity, where the diffusion time is much longer than the decoherence time. This means that the correlations are initially created at a long wavelength, and then are transported to the short-wavelength region by the decoherence and finally dissipated by the diffusion process. The presence of such an "inertial" (nondissipative) range is manifested in the long "tail" of $1/q$ dependence in the the correlator, where $q = J - \tilde{J}$.

The correlation function formalism that we introduced is based on the ensemble of different realizations of the driving forces. It appears reasonable to assume that the system under consideration possesses the usual ergodicity property and the same correlators would emerge when averaging over time on a single realization of the process. The proof of that however is not trivial and should be addressed in the future.

ACKNOWLEDGMENTS

The Fermi National Accelerator Laboratory is operated by the Universities Research Association, Inc. under contract with the U. S. Dept. of Energy. The work is supported in part by grants from NSF PHY-9221402 and the DOE DE-FG02-93 ER40801.

APPENDIX A: EVALUATION OF MOMENTS

In this Appendix, we calculate the moments of Δx's that enter the evolution Eqs. (12) and (13) by employing the conventional methods of stochastic differential equations [1]. We start with using the Hamilton equations of motion for the Hamiltonian (6) to present the increments ΔJ and $\Delta\Psi$ in the form

$$\Delta\Psi = \omega(J)\Delta t + \Delta\Psi_1 + \Delta\Psi_2,$$

$$\Delta J = \Delta J_1 + \Delta J_2, \quad (A1)$$

where ΔJ_1, $\Delta\Psi_1$ are the first-order terms,

$$\Delta J_1 = -\frac{\partial V}{\partial \Psi}\int_t^{t+\Delta t} dt'\ \xi(t'),$$
$$\Delta \Psi_1 = \frac{\partial V}{\partial J}\int_t^{t+\Delta t} dt'\ \xi(t'), \tag{A2}$$

and ΔJ_2, $\Delta\Psi_2$ are the second-order ones,

$$\Delta J_2 = -\left(\frac{\partial^2 V}{\partial \Psi^2}\frac{\partial V}{\partial J} - \frac{\partial^2 V}{\partial\Psi\partial J}\frac{\partial V}{\partial \Psi}\right)\int_t^{t+\Delta t} dt' \times \int_t^{t'} dt''\ \xi(t')\xi(t''),$$
$$\Delta \Psi_2 = \left(\frac{\partial^2 V}{\partial J\partial\Psi}\frac{\partial V}{\partial J} - \frac{\partial^2 V}{\partial J^2}\frac{\partial V}{\partial \Psi}\right)\int_t^{t+\Delta t} dt' \times \int_t^{t'} dt'\ dt''\ \xi(t')\xi(t''). \tag{A3}$$

Averaging over the δ-correlated random process ξ yields then the following expressions for the same-point moments of Δx's (no mixing of 1 and 2 variables),

$$\langle\Delta J\rangle = \langle\Delta J_2\rangle = -\frac{\Delta t}{2}\left(\frac{\partial^2 V}{\partial \Psi^2}\frac{\partial V}{\partial J} - \frac{\partial^2 V}{\partial\Psi\partial J}\frac{\partial V}{\partial \Psi}\right),$$

$$\langle\Delta\Psi\rangle = \omega(J)\Delta t + \langle\Delta\Psi_2\rangle = \omega(J)\Delta t + \frac{\Delta t}{2}\left(\frac{\partial^2 V}{\partial J\partial\Psi}\frac{\partial V}{\partial J} - \frac{\partial^2 V}{\partial J^2}\frac{\partial V}{\partial \Psi}\right),$$

$$\langle(\Delta J)^2\rangle = \langle\Delta J_1^2\rangle = \Delta t\left(\frac{\partial V}{\partial\Psi}\right)^2, \tag{A4}$$

$$\langle(\Delta \Psi)^2\rangle = \langle\Delta \Psi_1^2\rangle = \Delta t\left(\frac{\partial V}{\partial J}\right)^2,$$

$$\langle\Delta J\Delta\Psi\rangle = \langle\Delta J_1\Delta\Psi_1\rangle = -\Delta t\frac{\partial V}{\partial\Psi}\frac{\partial V}{\partial J},$$

and for the different-point moments,

$$\langle\Delta J\Delta\tilde J\rangle = \langle\Delta J_1\Delta\tilde J_1\rangle = \Delta t\frac{\partial V}{\partial\Psi}\frac{\partial\tilde V}{\partial\tilde\Psi},$$
$$\langle\Delta J\Delta\tilde\Psi\rangle = \langle\Delta J_1\Delta\tilde\Psi_1\rangle = -\Delta t\frac{\partial V}{\partial\Psi}\frac{\partial\tilde V}{\partial\tilde J},$$
$$\langle\Delta\Psi\Delta\tilde J\rangle = \langle\Delta\Psi_1\Delta\tilde J_1\rangle = -\Delta t\frac{\partial V}{\partial J}\frac{\partial\tilde V}{\partial\tilde\Psi}, \tag{A5}$$
$$\langle\Delta\Psi\Delta\tilde\Psi\rangle = \langle\Delta\Psi_1\Delta\tilde\Psi_1\rangle = \Delta t\frac{\partial V}{\partial J}\frac{\partial\tilde V}{\partial\tilde J}.$$

In the latter expressions, we used the notations $V = V(J,\Psi)$ and $\tilde V = V(\tilde J, \tilde\Psi)$.

In the small noise and fast oscillations approximation, the moments have to be averaged over the phase Ψ while keeping the dependence on the phase difference $\varphi = \Psi - \tilde\Psi$. Using the Fourier series $V(J,\Psi) = \sum_n V_n(J)e^{in\Psi}$, one obtains

$$\langle\!\langle\Delta J\rangle\!\rangle = \frac{1}{2}\Delta t\sum_n n^2\frac{\partial|V_n|^2}{\partial J},$$
$$\langle\!\langle\Delta \Psi\rangle\!\rangle = \omega(J)\Delta t,$$
$$\langle\!\langle(\Delta J)^2\rangle\!\rangle = \Delta t\sum_n n^2|V_n|^2, \tag{A6}$$
$$\langle\!\langle(\Delta \Psi)^2\rangle\!\rangle = \Delta t\sum_n \left|\frac{\partial V_n}{\partial J}\right|^2,$$
$$\langle\!\langle\Delta J\Delta\Psi\rangle\!\rangle = 0.$$

Note here that the relation $\langle\!\langle\Delta J\rangle\!\rangle = \frac{1}{2}\frac{\partial}{\partial J}\langle\!\langle(\Delta J)^2\rangle\!\rangle$ was verified experimentally [5] and was proven in general for all Hamiltonian systems with random noise [2,3]. Similarly, the phase averaging for the different-point moments yields the expressions

$$\langle\!\langle\Delta J\Delta\tilde J\rangle\!\rangle = \Delta t\sum_n n^2 V_n\tilde V_{-n}e^{in\varphi},$$
$$\langle\!\langle\Delta J\Delta\tilde \Psi\rangle\!\rangle = 0,$$
$$\langle\!\langle\Delta \Psi\Delta\tilde J\rangle\!\rangle = 0, \tag{A7}$$
$$\langle\!\langle\Delta\Psi\Delta\tilde\Psi\rangle\!\rangle = \Delta t\sum_n \frac{\partial V_n}{\partial J}\frac{\partial\tilde V_{-n}}{\partial\tilde J}e^{in\varphi}.$$

APPENDIX B: SOLVING THE EVOLUTION EQUATION FOR THE SPECTRUM

The evolution equation for the spectrum $\tilde K$ is obtained from the evolution equation (20) for the correlator K to be

$$i\alpha\frac{\partial^2\tilde K}{\partial k\partial\varphi} + [2D - F(\varphi)]k^2\tilde K = cF(\varphi)\delta(k). \tag{B1}$$

The solution of the homogeneous equation, which is periodic in φ, can be obtained by applying the separation of variables technique. The solution, $\tilde K_h(k,\varphi) = \tilde T(k,\varphi)$, is a linear combination of the nonorthogonal eigenmodes given by

$$\tilde T(k,\varphi) = \sum_{n\neq 0} B_n\tilde T_n(\varphi)e^{-\frac{2Dk^3}{3n\alpha}}, \tag{B2}$$

where B_n are arbitrary constants and

$$\tilde T_n(\varphi) = \exp\left\{-in\left(\varphi - \frac{1}{2D}\int_0^\varphi d\varphi_1\ F(\varphi_1)\right)\right\}. \tag{B3}$$

Similarly, the solution of the inhomogeneous equation (B1) is given by

$$\tilde K(k,\varphi) = \begin{cases} 0, & \text{if } nk\alpha < 0, \\ \tilde T(k,\varphi), & \text{otherwise,}\end{cases} \tag{B4}$$

with the boundary condition,

$$i\alpha\frac{d\tilde T(k=0,\varphi)}{d\varphi} = cF(\varphi). \tag{B5}$$

By changing the variable φ to the variable

$$\varphi' = \varphi - \frac{1}{2D}\int_0^{\varphi} d\varphi_1\, F(\varphi_1), \tag{B6}$$

the coefficients B_n for $n \neq 0$ can be obtained as

$$B_n = \frac{c}{2\pi n\alpha}\int_0^{2\pi} d\varphi\, F(\varphi) \times \exp\left\{ in\left(\varphi - \frac{1}{2D}\int_0^{\varphi} d\varphi_1\, F(\varphi_1)\right)\right\}. \tag{B7}$$

The coefficient $B_{0,\mathrm{inh}}$ is zero.

[1] K. Gardiner, *Handbook of Stochastic Methods* (Springer, Berlin, 1985); G. E. Ulenbeck and L. S. Ornstein, Phys. Rev. **36**, 823 (1930).

[2] G. Dôme, CERN Report No. 84-15, 1984 (unpublished), pp. 215–260.

[3] S. Krinsky and J. M. Wang, Part. Accel. **12**, 107 (1982).

[4] J. Ellison, B. S. Newberger, and H.-J. Shih, in *Proceedings of the 1991 IEEE Particle Accelerator Conference*, edited by L. Lizama and J. Chew (San Francisco, 1991), p. 216; H. J. Shih, J. Ellison, B. Newberger, and R. Cogburn, Superconducting Super Collider Report No. SSCL-578, 1992 (unpublished).

[5] S. Hansen *et al.*, IEEE Trans. Nucl. Sci. **24**, 1452 (1977); D. Boussard, G. Dome, and C. Graziani, in *Proceedings of the 11th Conference on High Energy Accelerators* (CERN, Geneva, 1980), p. 620.

[6] D. Boussard, Proceedings of the CERN Accelerator School, Oxford, 1985 [CERN Report No. 87-03 (unpublished)], Vol. 2, p. 416.